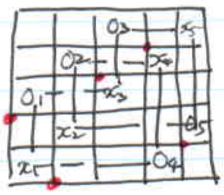
Manolescu

① invariance of HFK (grid) under commutations

② equivalence : pseudo holo. disks $\longleftrightarrow$ rectangles on a grid



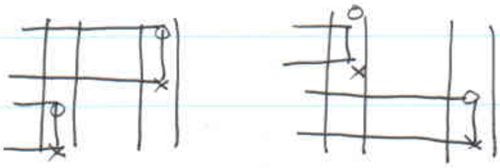
G : grid for a knot K

$S(G) = \{ g$ tuples of $\bullet \} \cong S_n$

$C(G) = \mathbb{Z}/2[U_1 \cdots U_n] \langle S(G) \rangle$

$\partial : C(G) \circlearrowleft$ $\qquad \partial \vec{x} = \sum_{\vec{y} \in S(G)} \sum_{r \in \mathrm{Rect}^\circ(\vec{x}, \vec{y})} U_1^{O_1(r)} \cdots U_n^{O_n(r)} \vec{y}$

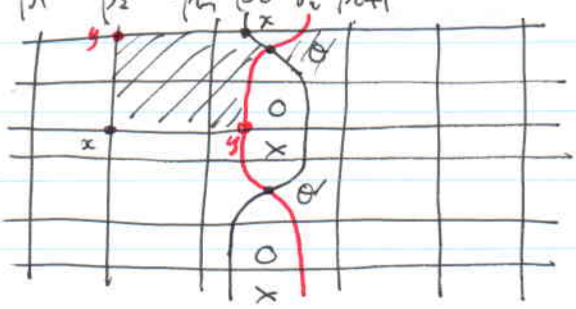
Commutation



with γ $\qquad$ with β

same knot

proof) CFK is invariant under commutation



change of grid

G = grid with α's and β's

H = grid with γ_i instead of β_i.

Prop. There are chain maps $\Phi_{\beta\gamma} : C(G) \longrightarrow C(H)$

$\Phi_{\gamma\beta} : C(H) \longrightarrow C(G)$

and $H_{\beta\gamma\beta} : C(G) \longrightarrow C(G)$

$H_{\gamma\beta\gamma} : C(H) \longrightarrow C(H)$

s.t. $\Phi_{\gamma\beta}\circ\Phi_{\beta\gamma} - I = \partial H_{\beta\gamma\beta} + H_{\beta\gamma\beta}\,\partial$

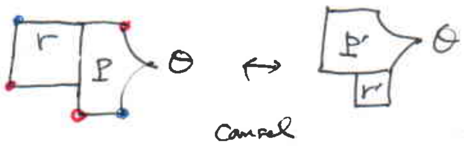
$\Phi_{\beta\gamma}\circ\Phi_{\gamma\beta} - I = \partial H_{\gamma\beta\gamma} + H_{\gamma\beta\gamma}\,\partial$

$\because$ $\underset{\substack{\cap \\ S(G)}}{\Phi_{\beta\gamma}(\vec{x})} = \displaystyle\sum_{y\in S(H)}\ \sum_{P\in \mathrm{Pent}^{\circ}(\vec{x},\vec{y})} u_1^{O_1(P)}\cdots u_r^{O_r(P)}\,\vec{y}$

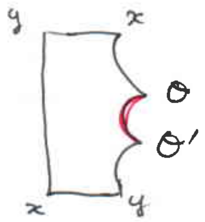
$\vdots$ with one vertex at Θ

$\Phi_{\gamma\beta}$ ... pentagon with one vertex at Θ'

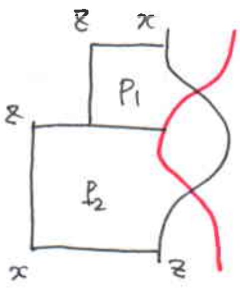
$\Phi_{\beta\gamma}\circ\partial = \partial\circ\Phi_{\beta\gamma}$

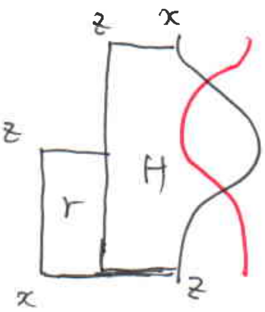


$H_{\beta\gamma\beta}$ = counts hexagons with vertices Θ & Θ'

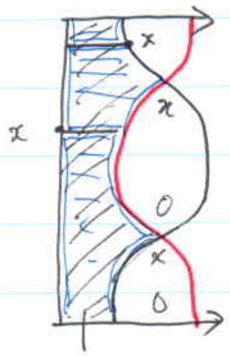


$\Phi_{\gamma\beta}\circ\Phi_{\beta\gamma} - I = H_{\beta\gamma\beta}\circ\partial + \partial\circ H_{\beta\gamma\beta}$





id $\vec{x} \to \vec{x}$ corresponds to



annuli

stabilization

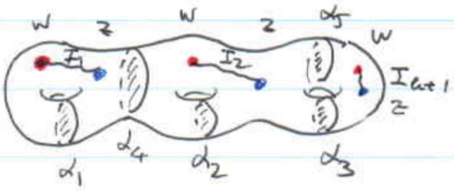
---- count of more complicated domains

but similar //

Part ② variation of def. of HFK using holo disks (Ozsváth-Szabó etc)

multi-pointed Heegaard diagram

$K \subset Y$ null hom. $\leadsto$ Σ genus g, $(\underbrace{\alpha_1, \cdots, \alpha_{g+k}}_{g\text{-dim sp. span } H_1}, \underbrace{\beta_1, \cdots, \beta_{g+k}}_{g\text{-dim sp.}})$



$w_1, \cdots, w_{k+1}$
$z_1, \cdots, z_{k+1}$

α's determine handlebody H

β's H'

$H \cup_\Sigma H' = Y^3$

Connect w to j by I_i's in the complement of α_i's

push to H

joint w to z in H' by J_i's

$$K = \bigcup I_i \cup J_{i'} \subset Y$$

$$\mathrm{Sym}^{g+k}(\Sigma)$$

$$\cup$$

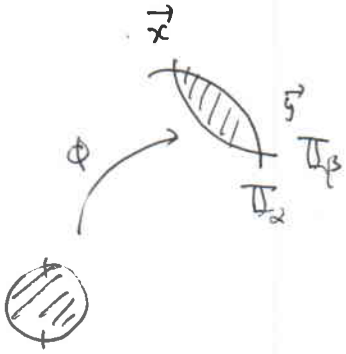
$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+k}$$

$$\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+k}$$

$$CF(\mathbb{T}_\alpha, \mathbb{T}_\beta) = \mathbb{Z}/2[U_1, \ldots, U_{k+1}]\langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle$$

$$\vec{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta = (x_{1\sigma(1)} \cdots x_{g+k,\sigma(g+k)}) \qquad x_{ij} \in \alpha_i \cap \beta_j$$

$$\partial \vec{x} = \sum_{\vec{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\vec{x}, \vec{y})} \# \left(\mathcal{M}(\phi)/\mathbb{R}\right) U_1^{n_1(\phi)} \cdots U_{k+1}^{n_{k+1}(\phi)} \vec{y}$$



$$n_i(\phi) = \left| u(\mathbb{D}^2) \cap \mathrm{Sym}^{g+k-1}(\Sigma) \times w_i \right|$$

Set $U_1 = 0 \Rightarrow \widehat{CFK}$

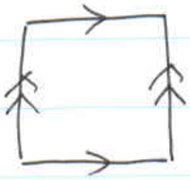
$U_1 = \cdots = U_{k+1} = 0 \Rightarrow \widetilde{CFK}$

$\longrightarrow$ HFK^-

$\widehat{HFK}$ Knot invariant

$\widetilde{HFK} \otimes V^k$

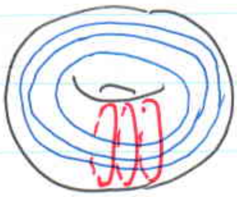
Example of multi-pointed Heegaard diagram for $K \subset S^3$



$\Sigma = T \quad (g=1)$

$\alpha_i = \{y = i\} \qquad i = 1, \dots, n \qquad n = k+1$

$\beta_j = \{x = j\}$

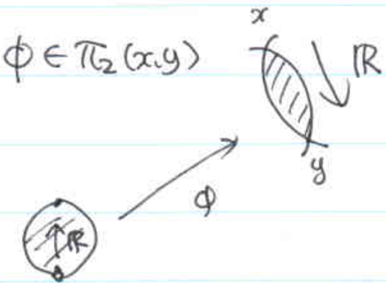


$O_i = w_i$

$X_i = z_i$

(index 1)
holo disk in $\mathrm{Sym}^n T$ $\longleftrightarrow$ rectangles in T
for a grid diagram

$\phi \in \pi_2(x, y)$



$\mathcal{M}(\phi)$ has an (expected) dimension
$= \mathrm{ind}(\phi) = $ Maslov index of ϕ
$\in \mathbb{Z}$

When $\mathrm{ind} = 1$, $\#(\mathcal{M}(\phi)/\mathbb{R})$ is well-defined.

tautological correspondence (Lipschitz) — Riemann surface

$$\left\{ u : \mathbb{D}^2 \xrightarrow[\text{hol}]{} \mathrm{Sym}^n \Sigma \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} S \xrightarrow{\text{hol.}} \Sigma \\ \downarrow {\scriptstyle n:1 \text{ branched cover}} \\ \mathbb{D}^2 \end{array} \right\} \qquad S \to \Sigma \times D^2$$

$S =$ pull back

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathrm{Sym}^{n-1}\Sigma \times \Sigma \arrow[d, "n:1"] \arrow[dr, "p_2"] & \\ D^2 \arrow[r, "u"'] & \mathrm{Sym}^n \Sigma & \Sigma \end{tikzcd}$$

$$\pi_1^{-1}(z) \longmapsto \pi_2(\pi_1^{-1}(z)) \in \mathrm{Sym}^n\Sigma$$

Conversely

$$\begin{array}{ccc} S & \xrightarrow{\pi_2} & \Sigma \\ \pi_1 \downarrow & & \\ z \in D & \longrightarrow & \mathrm{Sym}^n\Sigma \end{array}$$

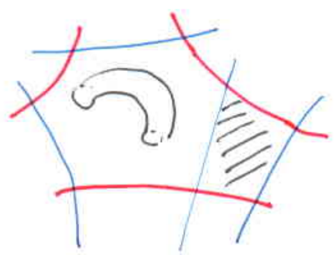
$u: \mathbb{D}^2 \to \mathrm{Sym}^n\Sigma$ define

Domain of $u \equiv D(u)$

$=$ Image of S in Σ.

α's and β's split Σ into some region $R_1, \dots, R_m$

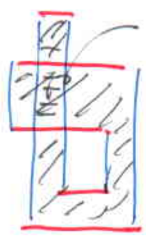
$$D(u) = \sum_{i=1}^{m} a_i R_i$$



$a_i \geq 0$

if u: holomorphic

Ex. On the grid



mult 2.

domain $\longleftrightarrow$ hom. class of disks $\pi_2(x, y)$ in $\mathrm{Sym}^n(D)$

index D = exp. dim of $\mathcal{M}(D)$

Claim ① $D = \square$ index 1 $\Longrightarrow$ $\#\mathcal{M}(D)/\mathbb{R}$ mod 1

② Only positive domains of index 1 on a grid are $\square$

about ①

$\tilde{S}$

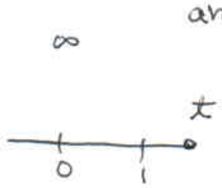
$\longrightarrow$

$\subset \Sigma$

$\downarrow 2:1$

Σ

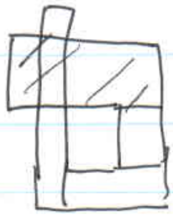
conformal structure on a disk u with 4 pts on ∂ are defined by cross-ratio



② index = combinatorial formula (due to Lipschitz)

index 1

index 2



index = 4